The Master Argument of Diodorus Cronus as an Alternative Account of Modality

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This paper considers an argument due to Diodorus Cronus (3rd century BCE), called by the ancients the Master Argument, which provides a theory of modality different from the modern orthodoxy of modal logic. It is argued that the Master Argument is supported by plausible intuitions, and that the modal logic obtained has reasonable epistemological motivation.

1. Introduction

Modal logic is the logic of possibility and necessity. We say that it is possible (though not actual) that snow had been green not white, and that it is necessary that 2+2=4. A simple and natural explication of these notions invokes the idea of possible worlds. This postulates that there are, in logical space, many worlds, all equally possible. One of these is the "real" world, ours. Possibility is explained as *truth in at least one possible world*, and necessity is explained as *truth in all possible worlds*; so that in some world other than our own snow really is green, while 2+2=4 in all worlds, it couldn't be otherwise. This idea goes back to Leibniz in the seventeenth century. In the late twentieth century, in the hands of the US philosopher David Lewis, it gave rise to a major industry. The movement was known as "modal realism", which means, in effect, that the theory postulates the existence of all these possible worlds (even thought we have no access to any world other than our own). Using this metaphysical machinery, Lewis and his followers offered solutions to many existing philosophical problems, such as the problem of the nature of counterfactuals, the nature of laws, personal identity and so on. The success of the program in being able to provide at least some answers to persisting difficult questions, is a reasonable argument in its favour, though not a decisive argument.

It must be realised, however, that modal realism faces a major objection, namely that since by definition there are no causal connections between possible worlds, and therefore no causal connections between our world and worlds other than our own, it is difficult to see how we could have any knowledge of them. Certainly, if we have intuitions about the truth of modal statements, these are not to be *explained* by the existence of worlds other than our own, since the existence of such worlds is causally irrelevant to producing such intuitions. But, if we could never know about them, there is no reason to believe in them. This difficulty is representative of a *kind* of objection, that *any* metaphysical position has to face: how to give a plausible *epistemology* for the metaphysical thesis; and so we can term it the *epistemological* objection. Other metaphysical items, such as numbers, or minds, face a similar objection, with varying degrees of success. The modal logic given by this natural but epistemically problematic modal realist account is called S5, which we will see more of below.

In this paper, we approach modal logic initially by laying down axioms, and then in terms of time and tense. Most importantly, we see that Diodorus Cronus (3rd century BCE) provided an ingenious argument, known as the Master Argument, for the conclusion that modality is best explicated in terms of time. Since it appeals to time, of which we plausibly have better knowledge than possible worlds, we can draw the reasonable conclusion that it is epistemically preferable as an account of modality. Further, we see that the logic of modality as so explained differs from S5.

We thus have an intelligible and epistemically more satisfactory alternative to the modern S5 orthodoxy of possible worlds, an alternative which modern modal logicians would do well to heed.

2. Three Modal Logics

Modern symbolic logic got underway with Gottlob Frege in the late nineteenth century. The axiomatic study of the nature of modality within symbolic logic got underway in the 1920s with Lewis and Langford. (This is C. I. Lewis, not David Lewis, who was no relation.) Lewis and Langford used axioms and rules to define a number of modal logics, of which we note two, S4 and S5. We insert a further important logic in between these two, S4.3, which was discovered later by Lemmon.

S4 (Lewis and Langford)

Extensional base: axioms and rules of two-valued Boolean logic with connectives &,v,~,→,↔ (we assume the reader is familiar with Boolean logic).

To this base, we add two modal symbols and their interpretations:

□p – It is necessary that p
◊p – It is possible that p

These concepts are interdefinable by:

Definitions (A): (Possibility and necessity)

□p := ~◊~p
◊p := ~□~p

Modal Axioms:

1. $\Box(p\to q)\to(\Box p\to\Box q)$
2. $\Box p\to p$ (equivalently, $p\to\Diamond p$)
3. $\Box p\to\Box\Box p$ (equivalently, $\Diamond\Diamond p\to\Diamond p$)

Modal Rule:

If any wff α is deducible, then we may deduce $\Box\alpha$ also

A proof (or valid deduction) is a sequence of lines which is either an axiom, or follows from earlier lines by the rules.

S4.3 (Lemmon)

Add to S4 any of the equivalent axioms:

4. $\Box(\Box p\to q)\vee\Box(\Box q\to p)$

4*. $\Box(\Box p\to\Box q)\vee\Box(\Box q\to\Box p)$

4**. $(\Diamond p\&\Diamond q)\to(\Diamond(p\&\Diamond q)\vee\Diamond(q\&\Diamond p))$

S5 (Lewis and Langford)

Add to S4 either of the equivalent axioms:

5. $p\to\Box\Diamond p$

5*. $\Diamond\Box p\to p$

Having defined these three logics by axioms and rules, it is obvious that S4.3 is at least as strong as S4, since the axioms and rules of S4 are included in those of S4.3. We also note:

Theorem 1 S4.3 is strictly stronger than S4. (Proof given later, in the appendix)

This then raises a question: Is S5 strictly stronger than S4.3? The answer is yes. To show this we must consider Diodorus Cronus modal logic, which develops a definition of necessity and possibility not in terms of worlds but in terms of tenses. If this can be made to work, then it is preferable to the account of modality given by modal realism, if only because tensed propositions are epistemically more accessible than possible worlds.

The logic of tenses is called, unsurprisingly, *tense logic*. The basic tenses studied in tense logic, together with their symbolic representations, are as follows:

Fp — to be interpreted as “future p” or “at some time in the future, p”
Gp — to be interpreted as “forever after p” or “at all times in the future, p”
Pp — to be interpreted as “past p” or “at some time in the past, p”
Hp — to be interpreted as “has always been that p” or “at all times in the past, p”

These are interdefinable by:

Definitions (B): (Past and Future)

$Fp := \sim G \sim p, \quad Gp := \sim F \sim p, \quad Pp := \sim H \sim p, \quad Hp := \sim P \sim p$

We now recall Aristotle's view that there are future contingents which are neither true nor false. But Diodorus Cronus (d.284BCE) disagreed with Aristotle on this point. Instead, he contended that the possible is what occurs now or in the future. The past is fixed, hence necessary. In support, Diodorus proposed what is known as the **Diodorean Master Argument.**

3. Diodorus Master Argument

Consider the three propositions:

(a) Every true proposition about the past is necessary.
(b) The impossible does not follow from the possible.
(c) Something that neither is nor will be, is possible.

Diodorus argues that these three are incompatible; and that hence, since (a) and (b) are true, (c) fails, so that the possible is exactly that which is either true now, or will be. Formally, these three propositions can be written:

(a) $Pp \rightarrow \Box Pp$
(b) $\Box(p \rightarrow q) \rightarrow (\sim\Diamond q \rightarrow \sim\Diamond p)$
(c) $\sim\pi \;\&\; \sim F\pi \;\&\; \Diamond\pi$ for some proposition π

It now has to be shown that the Master Argument has some plausible force. I claim that it is indeed plausibly motivated, and moreover (given two additional premisses) sound.

In motivating the premisses of the Master Argument, we note that (a) is a characteristic ancient thesis: since the past is fixed, it cannot be changed and so the proposition that it is past, is necessary. Arthur Prior says that both (a) and (b) were generally admitted (*Past Present and Future*, Oxford, Oxford UP 1967, 32).This explains why the Master Argument was taken seriously by the ancient logicians.

However, Prior also says that two further premisses are necessary to get the contradiction. These are:

(d) It is a necessary truth that if p is true (now) then it has always been that it will be true.
That is, formally: $\Box(p \rightarrow HFp)$

(e) If p is false and always will be false, then it has been that it will always be false.
That is, formally: $(\sim p \& G \sim p) \rightarrow PG \sim p$

Now, the additional premiss (d) of the Master Argument is also intuitively reasonable, and seems to have had the support of Aristotle *De Interpretatione* Chapter 9, and Cicero *De Fato*.

Thus the question of the plausible motivation of the Master Argument depends on the additional premiss (e), which deserves a comment. It would seem to be problematic if time is *densely* structured, that is, between any two temporal instants there is a third. After all, if now is the first instant when p becomes false and remains false hereafter, then the antecedent of (e) is true now, and yet any instant earlier than now has instants in its future, between it and now, at which p is true. This defeats G~p at any instant earlier than now, which in turn defeats PG~p now.

But the matter isn't quite so simple. If time is discrete (that is, non-dense), then at the instant before now, G~p holds, so that now PG~p holds as required. Of course, it would not be so surprising if the ancients did not have a clear concept of the denseness of time. I further suggest here that, in light of the quantum theory, it remains an open question whether time is discrete or dense.

Now with these five propositions we have:

Theorem 2 (The Master Argument). The propositions (a) – (e) are collectively inconsistent (for proof see Appendix).

We proceed then to Diodorus' account of modality, as motivated by the Master Argument. In light of the Master Argument, we can write:

Definition (C) (Diodorean Modality)

$\Diamond p := (p \vee Fp)$

My claim is that this gives, within the resources of tense logic, a non-S5 modality. To see this, we must first display, using the symbols of Definitions (B), two tense logics, basic minimal tense logic K_t, and an important extension NC. The extension is necessary to be able to demonstrate S4.3, but it has reasonable principles about tense which should be agreed by all parties.

4. Two Tense Logics

K_t (Lemmon)

Axioms. Basic Boolean logic plus:

6. $G(p \to q) \to (Fp \to Fq)$
7. $H(p \to q) \to (Pp \to Pq)$
8. $p \to GPp$
9. $p \to HFp$

Tense Rule:

If any wff α is provable, we may deduce $G\alpha$ and $H\alpha$

NC (Cocchiarella)

Add the axioms:

10. $FFp \to Fp$
11. $PPp \to Pp$

12. (Fp&Fq)→(F(p&q)vF(p&Fq)vF(q&Fp))
13. (Pp&Pq)→(P(p&q)vP(p&Pq)vP(q&Pp))

We now claim (for proofs see Appendix):

Theorem 3: Where possibility is defined as indicated by Diodorus Master Argument, (see Definition (C) above), all of S4.3 can be proved from NC, and

Theorem 4: The S5 axiom fails in this system.

These results thus provide us with the promised well-defined account of modality weaker than S5.

5. Conclusion

We conclude, as promised, that S4.3 is a well-motivated modal system, distinct from S5, epistemically preferable to modal realism, and supported by Diodorus Master Argument.

This prompts a concluding question about the role of formal systems in explaining and understanding informal concepts. Prior held that logic provides a formal framework in which metaphysical disputes can be conducted. Is what we have here a metaphysical dispute between two rival conceptions of modality, or is it simply different subject matters, to be classified alongside one another as different but compatible concepts?

6. Appendix: Sketch Proofs of the Main Theorems

In this appendix we sketch proofs of the main theorems. Basic familiarity with modal semantics is assumed. Those not familiar with modal semantics may skip the appendix.

Theorem 1: S4.3 is strictly stronger than S4.

Proof (Sketch): This can be seen by taking an accessibility relation (on possible worlds) which branches, which can be seen to verify S4 but falsifies S4.3.

Consider a normal modal model with three worlds w1, w2, w3, where w1 is the real world. The accessibility relation is defined to be w1Rw2 and w1Rw3, as well as each world being R to itself, that is w1Rw1, w2Rw2 and w3Rw3. It is a straightforward argument to verify that all theorems of S4 are T at w1. Finally, set p=T and q=F at w2, and p=F and q=T at w3 (values of p and q at w1 are irrelevant). It is then easy to see that □p=T at w2, so that □p→q is F at w2, and hence □(□p→q) is F at w1. By a similar argument, □q→p is F at w3, so that □(□q→p) is F at w1. Thus the S4.3 axiom □(□p→q)v□(□q→p) fails at the real world w1, while all theorems of S4 hold at w1, so that the S4.3 axiom is not provable in S4. QED

Theorem 2. The propositions (a) — (e) in Section 3 above are collectively inconsistent.

Proof: From $\sim\pi \& \sim F\pi$ by Definitions (B) and premiss (e), we have $PG\sim\pi$. Hence by (a), $\Box PG\sim\pi$. But, from (d) contraposing, $\Box(PG\sim\pi\rightarrow\sim\pi)$. Now it can be shown that (b) is equivalent to $\Box(p\rightarrow q)\rightarrow(\Box p\rightarrow\Box q)$. Hence, applying this to the previous two formulae, we have $\Box\sim\pi$. But from Definitions (A), this is equivalent to $\sim\Diamond\pi$, which contradicts the third conjunct of (c).

Theorem 3: Where possibility is defined as indicated by Diodorus Master Argument, (see (C) above), all of S4.3 can be proved from NC.

Proof: This is a matter of tracking through the axioms 1-4 and the Modal Rule, to verify that their translations under Definition (C) can all be proved in NC. We should also check that the premisses (a) and (b) of the Master Argument are also provable.

Axiom 1. $((p\rightarrow q)\&G(p\rightarrow q))\rightarrow((p\&Gp)\rightarrow(q\&Gq))$. It can be proved in K_t that $G(p\rightarrow q)\rightarrow(Gp\rightarrow Gq)$, which ensures the consequent given the antecedent.

Axiom 2. $(p\&Gp)\rightarrow p$. In basic Boolean logic, a conjunction implies each of its conjuncts.

Axiom 3. $(p\&Gp)\rightarrow((p\&Gp)\&G(p\&Gp))$. It can be proved in K_t that $Gp\rightarrow GGp$, and that $G(p\&Gp)$ is equivalent to $Gp\&GGp$. Axiom 3 follows.

Axiom 4**. $((p v Fp)\&(q v Fq))\rightarrow((p\&(q v Fq)) v F(p\&(q v Fq)) v (q\&(p v Fp)) v F(q\&(p v Fp))$.

Distributing the antecedent gives $(p\&q) v (p\&Fq) v (Fp\&q) v (Fp\&Fq)$. The first disjunct of the antecedent implies the first disjunct of the consequent, the second disjunct of the antecedent implies the first disjunct of the consequent, the third disjunct of the antecedent implies the third disjunct of the consequent. By Axiom 12, the fourth disjunct of the antecedent implies $F(p\&q) v F(p\&Fq) v F(q\&Fp)$. But $F(p\&q)$ implies $F(p\&(q v Fq))$ which is the second disjunct of the consequent; and $F(p\&Fq)$ implies $F(p\&(q v Fq))$ which is likewise the second disjunct of the consequent; and $F(q\&Fp)$ implies $F(q\&(p v Fp))$ which is the fourth disjunct of the consequent.

Modal Rule. From the Tense Rule, if α is provable, then $G\alpha$ is provable, so that $\alpha\&G\alpha$ is provable, which is $\Box\alpha$.

Premiss (a) of the Master Argument. $Pp\rightarrow(Pp\&GPp)$.From Axiom 8, $Pp\rightarrow GPPp$. But from Axiom 11, it is a theorem that $PP\rightarrow P$, so from the Tense Rule $G(PP\rightarrow P)$. In K_t the G distributes to give $GPP\rightarrow GP$, so applying this to $P\rightarrow GPP$ gives $Pp\rightarrow GPp$, from which Premiss (a) follows by Boolean principles.

Premiss (b) of the Master Argument. This is equivalent to:

> $((p\rightarrow q)\&G(p\rightarrow q))\rightarrow((p v Fp)\rightarrow(q v Fq))$. From $p\rightarrow q$ and p, the first disjunct of $p v Fp$, q follows so that $q v Fq$ follows. From $G(p\rightarrow q)$ and Fp, the second disjunct of $p v Fp$, Fq follows by Axiom 6, so that $q v Fq$ again follows.

Theorem 4: The S5 axiom fails in this system.

Proof: To see that the S5 Axiom 5 fails in Diodorus modal logic, we want to make $p \rightarrow \Box\Diamond p$ false now, that is $p \rightarrow ((p v Fp) \& G(p v Fp))$ fails to hold now. Take a model with two times, now and one in the future, τ. Set p=T now and p=F at τ. Then pvFp fails at τ, so that G(pvFp) fails now, hence $p \rightarrow (p v Fp) \& G(p v Fp)$ fails now.

Bibliography

Prior, 1967

Arthur Prior, *Past, Present and Future*. Oxford: Oxford UP.